7-2 First-Order Circuit Step Response

Linear circuits are often characterized by applying step function and sinusoid inputs. This section introduces the step response of first-order circuits. Later in this chapter we treat the sinusoidal response of first-order circuits and step response of second-order circuits. The step response analysis introduces the concepts of forced, natural, and zero-state responses that appear extensively in later chapters.

Our development of first-order step response treats the RC circuit in detail and then summarizes the corresponding results for its dual, the RL circuit. When the input to the RC circuit in Figure 7-2 is a step function, we can write the Thévenin source as \( v_T(t) = V_A u(t) \). The circuit differential equation in Eq. (7-13) becomes

\[
R_C \frac{dv}{dt} + v = V_A u(t) \tag{7-15}
\]

The step response is a function \( v(t) \) that satisfies this differential equation for \( t \geq 0 \) and meets the initial condition \( v(0) \). Since \( u(t) = 1 \) for \( t \geq 0 \) we can write Eq. (7-15) as
Mathematics provides a number of approaches to solving this equation, including separation of variables and integrating factors. However, because the circuit is linear we chose a method that uses superposition to divide solution \( v(t) \) into two components:

\[
v(t) = v_n(t) + v_f(t)
\]  

(7-11)

The first component \( v_n(t) \) is the natural response and is the general solution of Eq. (7-16) when the input is set to zero. The natural response has its origin in the physical characteristic of the circuit and does not depend on the form of the input. The component \( v_f(t) \) is the forced response and is a particular solution of Eq. (7-16) when the input is the step function. We call this the forced response because it represents what the circuit is compelled to do by the form of the input.

Finding the natural response requires the general solution of Eq. (7-16) with the input set to zero:

\[
R_C \frac{d v_n(t)}{d t} + v_n(t) = 0 
\]

(7-11)

But this is the homogeneous equation that produces the zero-input response in Eq. (7-8). Therefore, we know that the natural response takes the form

\[
v_n(t) = Ke^{-t/R_C} \quad t \geq 0
\]  

(7-11)

This is a general solution of the homogeneous equation because it contains an arbitrary constant \( K \). At this point we cannot evaluate \( K \) from the initial condition, as we did for the zero-input response. The initial condition applies to the total response (natural plus forced), and we have yet to find the forced response.

Turning now to the forced response, we seek a particular solution of the equation

\[
R_C \frac{d v_f(t)}{d t} + v_f(t) = V_A \quad t \geq 0
\]  

(7-11)

The equation requires that a linear combination of \( v_f(t) \) and its derivative equal a constant \( V_A \) for \( t \geq 0 \). Setting \( v_f(t) = V_A \) meets this condition since \( \frac{d v_f}{d t} = \frac{d V_A}{d t} = 0 \). Substituting \( v_f = V_A \) into Eq. (7-19) reduces it to the identity \( V_A = V_A \).

Now combining the forced and natural responses, we obtain

\[
v(t) = v_n(t) + v_f(t) \]

\[
= Ke^{-t/R_C} + V_A \quad t \geq 0
\]

This equation is the general solution for the step response because it satisfies Eq. (7-16) and contains an arbitrary constant \( K \). This constant can now be evaluated using the initial condition:

\[
v(0) = V_0 = Ke^0 + V_A = K + V_A
\]

The initial condition requires that \( K = (V_0 - V_A) \). Substituting this conclusion into the general solution yields the step response of the RC circuit.
A typical plot of \( v(t) \) is shown in Figure 7–12.

![Graph of \( v(t) \)](image)

The \( RC \) circuit step response in Eq. (7–20) starts out at the initial condition \( V_0 \) and is driven to a final condition \( V_A \), which is determined by the amplitude of the step function input. That is, the initial and final values of the response are

\[
\lim_{t \to 0^+} v(t) = (V_0 - V_A)e^{-R_IC} + V_A = V_0
\]

\[
\lim_{t \to \infty} v(t) = (V_0 - V_A)e^{-\infty} + V_A = V_A
\]

The path between the two end points is an exponential waveform whose time constant is the circuit time constant. We know from our study of exponential signals that the step response will reach its final value after about five time constants. In other words, after about five time constants the natural response decays to zero and we are left with a constant forced response caused by the step function input.